

Non-Hermitian Hamilton operator in open quantum systems

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Abstract

A powerful method for the description of open quantum systems is the Feshbach projection operator (FPO) technique. In this formalism, the whole function space is divided into two subspaces that are coupled with one another. One of the subspaces contains the wave functions localized in a certain finite region while the continuum of extended scattering wave functions is involved in the other subspace. The Hamilton operator of the whole system is Hermitian, that of the localized part is, however, non-Hermitian. This non-Hermitian Hamilton operator H_{eff} represents the core of the FPO method in present-day studies. It gives a unified description of discrete and resonance states. Furthermore, it contains the time operator. The eigenvalues z_λ and eigenfunctions ϕ_λ of H_{eff} are an important ingredient of the S matrix. They are energy dependent. The phases of the ϕ_λ are, generally, nonrigid. Most interesting physical effects are caused by the branch points in the complex plane. On the one hand, they cause the avoided level crossings that appear as level repulsion or widths bifurcation in approaching the branch points under different conditions. On the other hand, observable values are usually enhanced and accelerated in the vicinity of the branch points. In most cases, the theory is time asymmetric. An exception are the \mathcal{PT} symmetric bound states in the continuum appearing in space symmetric systems due to the avoided level crossing phenomenon in the complex plane. In the paper, the peculiarities of the FPO method are considered and three typical phenomena are sketched: (i) the unified description of decay and scattering processes, (ii) the appearance of bound states in the continuum and (iii) the spectroscopic reordering processes characteristic of the regime with overlapping resonances.

I. INTRODUCTION

An exact description of open quantum systems meets the problem to consider simultaneously the wave functions of discrete and scattering states. Both types of wave functions are completely different from one another. The discrete states λ characterize the spectrum of the system and are normalized according to the Kronecker delta $\delta_{\lambda\lambda'}$ while the scattering states are continuous in energy E and can be normalized according to the Dirac delta function $\delta(E - E')$. The wave functions of discrete and scattering states appear in a combined manner in most physical expressions characteristic of open quantum systems. Special mathematical considerations are necessary therefore in order to receive physical values.

In the N -level Friedrichs model [1, 2], the total Hamiltonian H is defined by

$$H = H_0 + \mu V \quad (1)$$

where μ is a real number and H_0 is the so-called free Hamiltonian

$$H_0 = \sum_n \omega_n |n\rangle \langle n| + \int_{K_\omega} \omega |\omega\rangle \langle \omega| \rho(\omega) d\omega. \quad (2)$$

Here, $|n\rangle$ and $|\omega\rangle$ satisfy the orthonormality condition $\langle n|n'\rangle = \delta_{nn'}$ and $\langle \omega|\omega'\rangle = \delta(\omega - \omega')/\rho(\omega)$, and $\langle n|\omega\rangle = 0$. The sum runs over the (finite) number of discrete basic states $|n\rangle$ and the integral is over the considered energy region with $K_\omega = \{\omega|\rho(\omega) \neq 0\}$. The interaction Hamilton operator V describes the coupling between $|n\rangle$ and $|\omega\rangle$,

$$V = \sum_n \int_{K_\omega} \left(v_n(\omega) |\omega\rangle \langle n| + v_n^*(\omega) |n\rangle \langle \omega| \right) \rho(\omega) d\omega \quad (3)$$

where $v_n(\omega)$ is the interaction matrix element between $|n\rangle$ and $|\omega\rangle$. In the Friedrichs model, the Schrödinger equation with the Hamiltonian H is directly solved. It is however not easy to receive results that are of physical interest in a broad range of parameters. For an example of the troubles see the study on bound states in the continuum [2].

Another method to solve the Schrödinger equation with the Hamilton operator (1) is the use of the Feshbach projection operator (FPO) technique. In this method, the basic equations for the wave functions of the states λ and ω are solved separately such that the main problem of the Friedrichs model is avoided. In the FPO formalism [3], the full function space is divided into two subspaces: the Q subspace contains all wave functions that are localized inside the system and vanish outside of it while the wave functions of the P subspace

are extended up to infinity and vanish inside the system, see [4]. The wave functions of the two subspaces can be obtained by standard methods: the Q subspace is described by the Hermitian Hamilton operator H_B that characterizes the closed system with discrete states, while the P subspace is described by the Hermitian Hamilton operator H_C that contains the continuum of scattering wave functions. Thus, $H_0 = H_B + H_C$ in (2). The coupling matrix elements are calculated according to (3) by using the eigenfunctions λ of H_B instead of the basic wave functions n that appear in (3). An example for the difference between the λ and the n is the following: in nuclear structure calculations, the n are determined by the Slater determinants while the λ are the shell-model wave functions. Furthermore, $\mu = 1$ in (1), i.e. there is no free parameter in H in the framework of the FPO formalism. All values are determined by fixing the potential and the coupling matrix elements between the different discrete and scattering states and by defining the two subspaces in a meaningful manner.

In the FPO formalism, the closed system (defined by the Hamilton operator H_B) will be opened by coupling the wave functions of the Q subspace to those of the P subspace under the assumption $P + Q = 1$. Due to this coupling, the discrete states of the closed system that lie above particle decay thresholds, pass into resonance states of the open system. The states below decay thresholds receive, as a rule, some energy shift but remain discrete. The resonance states have, in general, a finite life time.

The FPO method is introduced by Feshbach [3] forty years ago in order to describe nuclear reactions. At that time it was impossible to perform all the calculations in the two subspaces as well as those for the coupling matrix V . Instead, Feshbach used statistical assumptions for the narrow states of the Q subspace (compound nucleus states) and treated exactly only the so-called direct (fast) reaction part. In this manner it was possible to formulate a unified description of nuclear reactions, i.e. of the fast direct nuclear reaction part and the much slower compound nuclear reaction part.

In the present-day calculations on the basis of the FPO method, all the calculations in the Q subspace are performed with the same accuracy as the calculations for the corresponding closed system ($Q = 1$). Also the coupling matrix elements $v_\lambda(\omega)$ are calculated. These calculations represent therefore a unified description of structure and reaction phenomena [4]. They allow to draw general conclusions on the behavior of open quantum systems under different conditions, i.e. by controlling them in a broad parameter range.

In the present paper, the FPO method is sketched (Sect. 2) and some basic peculiarities

are discussed (Sect. 3). An important feature of the method is the appearance of the non-Hermitian Hamilton operator H_{eff} in an intermediate stage of the model. This Hamilton operator allows a unified description of discrete and resonance states. The complex eigenvalues z_λ and eigenfunctions ϕ_λ of H_{eff} determine decisively the S matrix (being unitary for all parameter values) and observables as will be shown by means of concrete examples in the following sections. Typical features of the formalism are time asymmetry (Sect. 4), the existence of bound (\mathcal{T} symmetric) states in the continuum when the system is \mathcal{P} symmetric (Sect. 5) and spectroscopic reordering processes that take place under the influence of branch points in the complex energy plane in the cross over from the weak-coupling regime to the strong-coupling one (Sect. 6).

II. BASIC RELATIONS OF THE FESHBACH PROJECTION OPERATOR (FPO) FORMALISM

The basic equation of the FPO formalism

$$(H - E) \Psi_C^E = 0 \quad (4)$$

is solved in the whole function space by dividing it into the two subspaces P and Q with $P + Q = 1$. The Hamilton operator H is Hermitian. It contains the decay of the subsystem localized in the Q subspace, into the surrounding P subspace where the decay products can be detected. The excitation of the states localized in the Q subspace may take place via one of the channels C included in (4) or by another process that can be described by a source term F appearing on the right-hand side of (4), for details see [5].

In solving (4) in the whole function space $P + Q = 1$ by using the FPO technique, the non-hermitian Hamilton operator

$$H_{\text{eff}} = H_B + \sum_C V_{BC} \frac{1}{E^+ - H_C} V_{CB} \quad (5)$$

appears which contains H_B as well as an additional non-hermitian term that describes the coupling of the resonance states via the common environment. Here V_{BC} , V_{CB} stand for the coupling matrix elements between the *eigenstates* of H_B and the environment [4] that may consist of different continua C . The operator H_{eff} characterizes the part of the problem that is localized in the Q subspace while the operator H describes the problem in the whole

function space $P + Q$. Therefore, H_{eff} is non-Hermitian and H is Hermitian. The operator H_{eff} is explicitly energy dependent and symmetric,

$$(H_{\text{eff}} - z_\lambda) \phi_\lambda = 0, \quad (6)$$

its eigenvalues z_λ and eigenfunctions ϕ_λ are complex. The eigenvalues provide not only the energies of the resonance states but also their widths. The eigenfunctions are biorthogonal. For details see [4].

The eigenvalues and eigenfunctions of H_B contain the interaction u of the discrete states which is given by the nondiagonal matrix elements of H_B . This interaction is of standard type in closed systems and may be called therefore internal interaction. The eigenvalues and eigenfunctions of H_{eff} contain additionally the interaction v of the resonance states via the common continuum (v is used here instead of the concrete matrix elements of the second term of H_{eff}). This part of interaction is, formally, of second order and may be called external interaction. While u and $\text{Re}(v)$ cause level repulsion in energy, $\text{Im}(v)$ is responsible for the bifurcation of the widths of the resonance states (resonance trapping). The phenomenon of widths bifurcation (resonance trapping) has been proven experimentally in microwave cavities [6].

Since the effective Hamilton operator (5) depends explicitly on energy E , so do its eigenvalues z_λ and eigenfunctions ϕ_λ . Far from thresholds, the energy dependence is weak, as a rule, in an energy interval of the order of magnitude of the width of the resonance state. The solutions of the fixed-point equations $E_\lambda = \text{Re}(z_\lambda)|_{E=E_\lambda}$ and of $\Gamma_\lambda = -2 \text{Im}(z_\lambda)|_{E=E_\lambda}$ are numbers that coincide (approximately) with the poles of the S matrix. In the FPO formalism, however, it is not necessary to consider the poles of the S matrix since the spectroscopic information on the system follows directly from the complex eigenvalues z_λ and eigenfunctions ϕ_λ of H_{eff} . Moreover, in the physical observables related to the S matrix the eigenvalues z_λ with their full energy dependence are involved, see (15). Due to this fact, information on the vicinity (in energy) of the considered resonance states such as the position of decay thresholds and of neighboring resonance states is involved in the S matrix and can be received. Such an information can not be obtained from the poles of the S matrix being (energy-independent) numbers.

In contrast to the trajectories $z_\lambda(X)$ of the eigenvalues of a Hermitian Hamilton operator (where X is a certain parameter), those of a non-Hermitian one may cross in the complex

plane. The crossing points are branch points (called exceptional points in the mathematical literature). Physically, they are responsible for the avoided level crossing phenomenon appearing in their vicinity. More precisely: in approaching the branch points under different conditions, we have level repulsion (together with widths equilibration) or widths bifurcation (together with level attraction), see [4].

The eigenfunctions ϕ_λ of H_{eff} are complex and biorthogonal. The normalization condition $\langle \phi_\lambda^{\text{left}} | \phi_\lambda^{\text{right}} \rangle = \langle \phi_\lambda^* | \phi_\lambda \rangle$ fixes only two of the four free parameters [7]. This freedom can be used in order to provide a smooth transition from an open quantum system (with, in general, nonvanishing decay widths Γ_λ of its states and biorthogonal wave functions ϕ_λ) to the corresponding closed one (with $\Gamma_\lambda \rightarrow 0$ and real wave functions that are normalized in the standard manner): $\langle \phi_\lambda^* | \phi_\lambda \rangle \rightarrow \langle \phi_\lambda | \phi_\lambda \rangle = 1$ if the coupling vectors in the non-Hermitian part of (5) vanish. That means, the orthonormality conditions can be chosen as

$$\langle \phi_\lambda^* | \phi_{\lambda'} \rangle = \delta_{\lambda, \lambda'} \quad (7)$$

with the consequence that [4]

$$\langle \phi_\lambda | \phi_\lambda \rangle \equiv A_\lambda \geq 1 \quad (8)$$

$$B_\lambda^{\lambda'} \equiv \langle \phi_\lambda | \phi_{\lambda' \neq \lambda} \rangle = -B_\lambda^{\lambda'} \equiv -\langle \phi_{\lambda' \neq \lambda} | \phi_\lambda \rangle; \quad |B_\lambda^{\lambda'}| \geq 0. \quad (9)$$

Approaching the branch point where the two eigenvalues z_λ and $z_{\lambda'}$ coalesce, $A_\lambda \rightarrow \infty$ and $|B_\lambda^{\lambda'}| \rightarrow \infty$. The normalization condition (7) entails that the phases of the eigenfunctions in the overlapping regime are not rigid: the normalization condition $\langle \phi_\lambda^* | \phi_\lambda \rangle = 1$ is fulfilled, in this regime, only when $\text{Im} \langle \phi_\lambda^* | \phi_\lambda \rangle \propto \text{Re} \phi_\lambda \cdot \text{Im} \phi_\lambda = 0$, i.e. by rotating the wave function at a certain angle β_λ . For details see [8]. The phase rigidity defined by

$$r_\lambda = \frac{\langle \phi_\lambda^* | \phi_\lambda \rangle}{\langle \phi_\lambda | \phi_\lambda \rangle} = \frac{1}{(\text{Re} \phi_\lambda)^2 + (\text{Im} \phi_\lambda)^2} = \frac{1}{A_\lambda} \quad (10)$$

is a useful measure [9] for the rotation angle β_λ . When the resonance states are distant from one another, it is $r_\lambda \approx 1$ due to $\langle \phi_\lambda | \phi_\lambda \rangle \approx \langle \phi_\lambda^* | \phi_\lambda \rangle$. In approaching a branch point in the complex energy plane [4, 10], we have $r_\lambda \rightarrow 0$. Therefore $1 \geq r_\lambda \geq 0$.

The phase rigidity r_λ is a measure for the degree of alignment of one of the overlapping resonance states with one of the scattering states ξ_C^E of the environment. This alignment takes place at the cost of the other states that decouple, to a certain extent, from the environment (*widths bifurcation* or *resonance trapping* [4]). It agrees with experimental data

[11], according to which the phase rigidity drops smoothly from its maximum value $r_\lambda = 1$ far from the branch point to its minimum value $r_\lambda = 0$ at the branch point, see [8].

The solution of (4) reads [4]

$$|\Psi_C^E\rangle = |\xi_C^E\rangle + \sum_\lambda |\Omega_\lambda^C\rangle \frac{\langle \phi_\lambda^* | V | \xi_C^E \rangle}{E - z_\lambda} \quad (11)$$

where

$$|\Omega_\lambda^C\rangle = \left(1 + \frac{1}{E^+ - H_C} V_{CB}\right) |\phi_\lambda\rangle \quad (12)$$

is the wave function of the resonance state λ and the ξ_C^E are the (coupled) scattering wave functions of the continuum into which the system is embedded. The expression (11) follows by applying $P + Q = 1$ to (4) without any approximations. It is therefore an exact solution of (4). The representation of Ψ_C^E (being solution of (4) with the Hermitian operator H) in the set of wave functions ϕ_λ (being solutions of (6) with the non-Hermitian operator H_{eff}) characterizes the consideration of localized states in the FPO formalism. According to (11), the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator H_{eff} give the main contribution to the wave function Ψ_C^E in the interior of the system,

$$|\Psi_C^E\rangle \rightarrow |\hat{\Psi}_C^E\rangle = \sum_\lambda c_{C\lambda}^E |\phi_\lambda\rangle; \quad c_{C\lambda}^E = \frac{\langle \phi_\lambda^* | V | \xi_C^E \rangle}{E - z_\lambda} \quad (13)$$

and

$$\langle \Psi_C^E | \rightarrow \langle \hat{\Psi}_C^E | = \sum_\lambda d_{C\lambda}^E \langle \phi_\lambda^{\text{left}} | = \sum_\lambda d_{C\lambda}^E \langle \phi_\lambda^* |. \quad (14)$$

The weight factors $c_{C\lambda}^E$ and $d_{C\lambda}^E$ ($= c_{C\lambda}^{E*}$ in the scattering process) contain decay and excitation, respectively, of the states λ at the energy E .

The S matrix is unitary. It can be obtained from $\langle \xi_C^E | V | \Psi_C^E \rangle$, see [4]. The amplitude of the resonance part we are interested in, is given by

$$S^{\text{res}} = i \sum_\lambda \langle \xi_C^E | V | \phi_\lambda \rangle c_{C\lambda}^E = i \sum_\lambda \frac{\langle \xi_C^E | V | \phi_\lambda \rangle \langle \phi_\lambda^* | V | \xi_C^E \rangle}{E - z_\lambda}. \quad (15)$$

The resonance structure of S^{res} is determined by the eigenvalues z_λ of H_{eff} as long as the resonances do not overlap (see Sect. 6).

III. PECULIARITIES OF THE FESHBACH PROJECTION OPERATOR (FPO) FORMALISM. UNIFIED DESCRIPTION OF DISCRETE AND RESONANCE STATES

The characteristic features of the FPO formalism consist, above all, in the fact that the solution Ψ_C^E in the whole function space can be represented in the set of wave functions $\{\phi_\lambda\}$ that describe the localized part of the problem. The localized wave functions represent a subspace of the whole function space with the consequence that the corresponding Hamilton operator H_{eff} is non-Hermitian.

The main advantages of the FPO formalism consist in the following.

(i) The spectroscopic information on the resonance states is obtained directly from the complex eigenvalues z_λ and eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator H_{eff} . The z_λ and ϕ_λ are energy dependent functions, generally, and contain the influence of neighboring resonance states as well as of decay thresholds onto the considered state λ . This energy dependence allows to describe decay and resonance phenomena also in the very neighborhood of decay thresholds and in the regime of overlapping resonances. Since also the coupling coefficients between system and continuum depend on energy, the unitarity of the S matrix is guaranteed for all parameter values, see e.g. [12].

(ii) The resonance states are directly related to the discrete states of a closed system described by standard quantum mechanics (with the Hermitian Hamilton operator H_B). They are generated by opening the system, i.e. by coupling the discrete states to the environment of scattering states by means of the second term of the Hamilton operator H_{eff} . Therefore, they are realistic localized (long-lived many-particle) states of an open quantum system.

(iii) The properties of branch points and their vicinity can be studied relatively easy. At these points, two (or more) eigenvalues z_λ of H_{eff} coalesce. Since it is not necessary to consider the poles of the S matrix in the FPO formalism, additional mathematical problems at and in the vicinity of branch points (exceptional points) in the complex plane are avoided.

(iv) The phases of the eigenfunctions ϕ_λ of H_{eff} are not rigid in the vicinity of a branch point. This fact allows to describe the spectroscopic reordering processes in the system that take place under the influence of the scattering wave functions ξ_C^E of the environment into which the system is embedded.

These features are involved in all present-day [13] calculations performed on the basis of the FPO formalism. In numerical studies, the main problem arises from the definition of the two subspaces Q and P such that it is meaningful for spectroscopic studies (the cross section is, of course, independent of the manner the two subspaces are defined, for more details see [4]). The basic idea is the following: H_B describes the closed system (localized in the interior of the system) which becomes open when embedded in the environment of scattering wave functions described by H_C . Therefore, all values characteristic of resonance states can be traced back to the corresponding values of discrete states by controlling the coupling to the continuum. That means with $v \rightarrow 0$, the transition from resonance states (described by the non-Hermitian H_{eff}) to discrete states (described by the Hermitian H_B) can be controlled.

As to the mathematical properties of branch points in the complex energy plane, it will be mentioned here only that the phase jump behavior of the wave functions at and in the vicinity of the branch points and their topological structure are investigated in the framework of the FPO formalism by means of numerical calculations for some special cases as well as in analytical studies [10, 14, 15, 16]. The topological structure of the branch (exceptional) points in the continuum differs from that of diabolic points. The results obtained in the FPO formalism are in full agreement with those of rigorous mathematical studies for a symmetric non-Hermitian 2x2 matrix Hamiltonian [7]. At the exceptional (branch) point, the two different right and left eigenvectors of the non-Hermitian Hamilton operator are linearly dependent, $|\phi_\lambda\rangle \leftrightarrow \pm i|\phi_{\lambda'}\rangle$; $\langle\phi_\lambda^*| \leftrightarrow \mp i\langle\phi_{\lambda'}^*|$, and are supplemented by the corresponding associated vectors defined by Jordan chain relations. Moreover, the results are in agreement with those of experimental studies on microwave cavities [17], see Ref. [8]. The cross section (S matrix) behaves smoothly at the branch point [4, 18].

The physical meaning of the branch points in the complex energy plane is based upon their topological structure and their relation to the phenomenon of avoided level crossing, i.e. to, respectively, level repulsion and widths bifurcation occurring in approaching them under different conditions. Level repulsion is accompanied by widths equilibration, while widths bifurcation occurs together with level attraction (formation of clusters). For further details see e.g. [4].

In the FPO formalism, a unified description of discrete and resonance states is involved. First, the resonance states pass smoothly into discrete states by reducing the coupling

strength v between system and environment. The smooth transition is guaranteed by choosing the orthonormality conditions for the eigenfunctions ϕ_λ of H_{eff} according to (7). Secondly, the localized states in the Q subspace are eigenstates of H_{eff} . Generally, they may lie above as well as below the particle decay thresholds. In the first case, their widths Γ_λ are different from zero (with the exception of bound states in the continuum, see Sect. 5) while they are zero in the second case. In the last case, H_{eff} is real: the residuum of the second term of (5) vanishes and only the principal value integral is different from zero. It causes, generally, some shift of the position of the state λ , $\Delta(E_\lambda^B - E_\lambda) \neq 0$, see [4]. Also the wave functions of these states differ, generally, from those of the eigenfunctions of H_B , $\phi_\lambda \neq \phi_\lambda^B$. That means, discrete and resonance states are described in a unified manner by the non-Hermitian Hamilton operator H_{eff} : not only the resonance states lying above particle decay thresholds, but also the discrete states below the thresholds are influenced by the continuum. As a consequence, the branch points in the complex plane determine also the properties of bound states. The most impressive example is the phenomenon of avoided crossing of discrete states known since the very beginning of quantum physics [19]. It can be traced from discrete states up to the branch points in the complex plane by parameter variation [15]. At the branch points, nonlinearities play a role. The branch points introduce therefore nonlinear effects into quantum mechanics [4]. This fact may lead to a deeper understanding of the relation between avoided level crossings and quantum chaos.

These results show that the non-Hermitian Hamilton operator H_{eff} is basic for the description of the localized states in realistic quantum systems. It describes the spectroscopic properties of discrete and resonance states in a unified manner. The restriction to the Hermitian Hamilton operator H_B [see (5)] in the standard quantum mechanics is an approximation: the principal value integral of the second term of (5) is effectively taken into account, however the residuum is neglected. Although this approximation works well in *very* many applications, it leaves open some fundamental questions. In these cases, the non-Hermiticity of the Hamilton operator H_{eff} (and the nonlinear effects related to it) can not be neglected.

Another peculiarity of the FPO formalism is the existence of a time operator which is the residuum of the non-Hermitian Hamilton operator H_{eff} . The life time τ_λ of a resonance state follows from the eigenvalue z_λ of H_{eff} in the same manner as the energy E_λ of this state. Both values are fundamentally different from the time t and the energy E . They characterize

the state λ while t and E appear as general parameters. In the closed system with the Hermitian Hamilton operator H_B , only the energies E_B of the states can be determined. The eigenvalues are real and the widths are zero, $\Gamma_B = 0$. Due to the coupling to the continuum, energy shifts $E_\lambda - E_B$ of the states appear as well as the finite life times $\tau_\lambda \propto (\Gamma_\lambda - \Gamma_B)^{-1} = \Gamma_\lambda^{-1}$ of the resonance states. Both, the energy shifts and the finite life times, follow from the non-Hermitian coupling term of H_{eff} [see Eq. (5)]. Usually, the numbers E_λ and Γ_λ can be obtained directly from the z_λ . Only in the case the z_λ are strongly dependent on energy, the corresponding fixed-point equations have to be solved. The energies E_λ and life times τ_λ of the resonance states λ of an open quantum system are bounded from below (see [8] for the discussion of the brachistochrone problem in open quantum systems). Mathematically, the existence of the time operator entails the time asymmetry involved in the FPO formalism.

IV. UNIFIED DESCRIPTION OF RESONANCE AND DECAY PHENOMENA

The time dependent Schrödinger equation reads

$$H_{\text{eff}} \hat{\Psi}^E(t) = i \hbar \frac{\partial}{\partial t} \hat{\Psi}^E(t). \quad (16)$$

The right solutions may be represented, according to (13), by an ensemble of resonance states λ that describes the decay of the localized part of the system at the energy E ,

$$\begin{aligned} |\hat{\Psi}^{E \text{ (right)}}(t)\rangle &= e^{-iH_{\text{eff}} t/\hbar} |\hat{\Psi}^{E \text{ (right)}}(t_0)\rangle \\ &= \sum_{\lambda} e^{-iz_{\lambda} t/\hbar} c_{\lambda 0} |\phi_{\lambda}^{(\text{right})}\rangle \end{aligned} \quad (17)$$

with $|\phi_{\lambda}^{(\text{right})}\rangle = |\phi_{\lambda}\rangle$ and $c_{\lambda 0} = \langle \phi_{\lambda}^* | V | \xi_C^E \rangle / (E - z_{\lambda})$. The z_{λ} and ϕ_{λ} are the (energy dependent) eigenvalues and eigenfunctions of the time-independent Hamilton operator H_{eff} , Eq. (5), while the ξ_C^E are the scattering wave functions of the environment. The left solution of (16) reads

$$\begin{aligned} \langle \hat{\Psi}^{E \text{ (left)}}(t) | &= \langle \hat{\Psi}^{E \text{ (left)}}(t_0) | e^{iH_{\text{eff}}^{\dagger} t/\hbar} \\ &= \sum_{\lambda} \langle \phi_{\lambda}^{(\text{left})} | d_{\lambda t} e^{iz_{\lambda}^* t/\hbar} \end{aligned} \quad (18)$$

with $\langle \phi_{\lambda}^{(\text{left})} | = \langle \phi_{\lambda}^* |$ and $d_{\lambda t} = c_{\lambda 0}^* = \langle \xi_C^E | V | \phi_{\lambda} \rangle / (E - z_{\lambda}^*)$ for the scattering process. It describes the excitation of the system at the energy E . For other excitation processes,

e.g. via a source term F on the right-hand side of (4), see [5]. Here $d_{\lambda t}$ is, generally, time dependent.

By means of (17) and (18) the population probability

$$\langle \hat{\Psi}_C^{E \text{ (left)}}(t) | \hat{\Psi}_C^{E \text{ (right)}}(t) \rangle = \sum_{\lambda} c_{\lambda 0} d_{\lambda t} e^{-\Gamma_{\lambda} t / \hbar} \quad (19)$$

at the energy E can be defined. The decay rate reads

$$\begin{aligned} k_{\text{gr}}(t) &= -\frac{\partial}{\partial t} \ln \langle \hat{\Psi}_C^{E \text{ (left)}}(t) | \hat{\Psi}_C^{E \text{ (right)}}(t) \rangle \\ &= \frac{1}{\hbar} \frac{\sum_{\lambda} \Gamma_{\lambda} c_{\lambda 0} d_{\lambda t} e^{-\Gamma_{\lambda} t / \hbar}}{\sum_{\lambda} c_{\lambda 0} d_{\lambda t} e^{-\Gamma_{\lambda} t / \hbar}}. \end{aligned} \quad (20)$$

For an isolated resonance state λ , (20) passes into the standard expression

$$k_{\text{gr}}(t) \rightarrow k_{\lambda} = \Gamma_{\lambda} / \hbar. \quad (21)$$

In this case, the value k_{λ} is constant in time and corresponds to the standard relation $\tau_{\lambda} = \hbar / \Gamma_{\lambda}$ with $\tau_{\lambda} = 1 / k_{\lambda}$. It describes the idealized case of an exponential decay law and, according to (15), a Breit-Wigner resonance in the cross section. Generally, deviations from the exponential decay law and from the Breit-Wigner line shape appear under the influence of neighboring resonance states and (or) of decay thresholds. Also the background term appearing in most reactions may cause deviations from the ideal exponential decay law. For details see [5]

The expressions (17) and (18) are valid only when (13) holds, i.e. at times t at which the wave functions Ψ_C^E have a localized part in the interior of the system at the energy E so that the representation (13) is meaningful at this energy. According to (17) and (18), this is the case for times $t \geq t_0$ where t_0 is a finite value. Without loss of generality, it can be chosen $t_0 = 0$. The quantum system described in the framework of the FPO formalism is therefore time asymmetric. The time asymmetry is involved in the non-Hermitian part of the Hamilton operator H_{eff} (which contains the time operator), as can be seen immediately from the expression (19) for the population probability.

The consideration of only the time interval $0 \leq t \leq \infty$ in (16) is related to the fact that the decay of a resonance state (at the energy E of the system) starts at a finite time (say $t_0 = 0$) at which the system can be considered to be excited, i.e. (13) is meaningful at this energy. This fact agrees with the concept of a semigroup description introduced in [20],

which distinguishes between prepared and measured states. In our formalism, the decaying (measured) states are described by the eigenvalues and eigenfunctions of the effective non-Hermitian Hamilton operator H_{eff} involved in the $|\hat{\Psi}^E\rangle$, Eq. (13). The preparation of the resonance states is described by the energy-dependent $\langle\hat{\Psi}^E|$. It may be very different for different reactions.

The decay properties of the resonance states can be studied best when their excitation takes place in a time interval that is very short as compared to the life time τ_λ of the resonance states, i.e. $d_{\lambda t}$ is a function being strongly dependent on time. In such a case, the time $t_0 = 0$ is well defined and no perturbation of the decay process by the still continuing excitation process will take place. In [21], such a situation is studied in single ion experiments. The results demonstrate the beginning of time for a decaying state. That means, they prove the time asymmetry in quantum physics.

Eq. (20) describes the decay rate also in the regime of overlapping resonances, see [5, 22]. The overlapping and mutual influence of resonance states is maximal at the branch points in the complex plane where two eigenvalues z_λ and $z_{\lambda'}$ of the effective Hamilton operator H_{eff} coalesce. Nevertheless, the decay rate is everywhere smooth as can be seen also directly from (20). This result coincides with the general statement according to which all observable quantities behave smoothly at singular points.

Another interesting problem is the saturation of the average decay rate k_{av} in the regime of strongly overlapping resonances. According to the bottle-neck picture of the transition state theory, it starts at a certain critical value of bound-continuum coupling [23]. This saturation is caused by widths bifurcation (formation of long-lived resonance states by resonance trapping by a few short-lived states [4]) occurring in the neighborhood of the branch points in the complex plane [8, 24]. Widths bifurcation creates long-lived resonance states together with a few short-lived resonance states. The definition of an average life time of the resonance states is meaningful therefore only for either the long-lived states or the short-lived ones. The long-lived (trapped) resonance states are almost decoupled from the continuum of decay channels. Their widths Γ_λ saturate with increasing bound-continuum coupling. The Γ_λ are almost the same for all the different states λ , see [4], i.e. $\Gamma_{\text{av}} \approx \Gamma_\lambda$ for all long-lived trapped resonance states. It follows therefore

$$k_{\text{av}} \approx \Gamma_{\text{av}}/\hbar \quad (22)$$

from (20). According to the average width Γ_{av} , the average life time of the long-lived states can be defined by $\tau_{\text{av}} = 1/k_{\text{av}}$. Then (22) is equivalent to $\tau_{\text{av}} = \hbar/\Gamma_{\text{av}}$. That means, the basic relation between life times and decay widths of resonance states holds not only for isolated resonance states [see Eq. (21)], but also for narrow resonance states superposed by a smooth background (that may originate from a few short-lived resonance states [4, 25]). In the last case, the relation holds for the average values Γ_{av} and τ_{av} .

The expressions (15) for the S matrix and (20) for the decay rate show immediately that the resonance phenomena (described by the S matrix) are determined by the decay properties of the resonance states (described by the complex eigenvalues z_λ and eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator H_{eff}). Thus, the FPO formalism provides a unified description of resonance and decay phenomena. The expression (15) shows however also that, generally, the energy dependence of the eigenvalues z_λ and eigenfunctions ϕ_λ of H_{eff} causes deviations from the Breit-Wigner resonance shape and the exponential decay law. The deviations become important for isolated resonance states due to the fact that the decay thresholds lie at a finite energy [26]. They appear mainly in the long-time scale. This result agrees qualitatively with experimental data [27]. At high level density, deviations appear even in the short-time scale due to the mutual influence of neighbored resonance states, see Sect. 5 and [8].

V. BOUND STATES IN THE CONTINUUM (BICS)

The question whether or not bound states in the continuum (BICs) exist in realistic quantum systems is of principal interest and might be as well of interest for applications. The reason for this interest arises from the fact that the system is stabilized at a BIC as well as in its vicinity, and that the wave function is localized at all times inside the system in spite of embedding it into the continuum of extended wave functions.

Mathematically, the existence of bound states in the continuum is shown already in 1929 by von Neumann and Wigner [28]. In 1985 Friedrich and Wintgen [29] considered the problem by using the FPO technique. They related the existence of BICs to avoided level crossings being another quantum mechanical phenomenon discussed by von Neumann and Wigner [19] in 1929. As discussed in Sects. 2 and 3, avoided level crossings are caused by branch points and appear in their vicinity.

Since BICs are states that do not decay, the population probability of these states is constant in time. This fact is called population trapping in studies on laser induced continuum structures in atoms [30]. Similar results are obtained [14] in the time independent approach by using the FPO technique and demanding a vanishing decay width for the BIC. In these papers, the relation between BICs and the avoided level crossing phenomenon as well as the stabilization of the system in a broad range of the parameter values (characteristic of the laser) is shown explicitly. A similar study is performed for the transmission through a quantum billiard where BICs appear at those energies at which the resonant transmission crosses a transmission zero [31, 32]. Common to all these studies is the definition of a BIC as a resonance state with vanishing width,

$$\Gamma_{\lambda_0}|_{(E=E_{\lambda_0})} = 0 . \quad (23)$$

In the FPO formalism, its energy is obtained from the solution of the fixed-point equation $E_{\lambda_0} = E_{\lambda}|_{(E=E_{\lambda_0})}$.

Generally, the relation between Γ_{λ} and the coupling matrix elements is [4]

$$\Gamma_{\lambda} = -2 \text{Im}(z_{\lambda}) \leq 2\pi \sum_C |\langle \phi_{\lambda}^* | V | \xi_C^E \rangle|^2 . \quad (24)$$

This expression holds true at all energies. The inequality in (24) is caused by the biorthogonality of the functions ϕ_{λ} being eigenfunctions of the non-Hermitian operator (5). As a consequence, a state being decoupled from all channels C of the continuum according to

$$\langle \xi_C^E | V | \phi_{\lambda_0} \rangle \rightarrow 0 \quad (25)$$

is a BIC with $\Gamma_{\lambda_0} \equiv -2 \text{Im}(z_{\lambda_0}) \rightarrow 0$ [the condition (25) is equivalent to $\langle \phi_{\lambda_0}^* | V | \xi_C^E \rangle \rightarrow 0$ due to the symmetry of H_{eff} and the biorthogonality of the ϕ_{λ}]. The opposite case follows by considering the S matrix, see (15) for the amplitude of its resonance part. At the position of a BIC, we have $E - z_{\lambda_0} \rightarrow 0$ and, due to the unitarity of the S matrix, it follows (25) for all C . That means: the decoupling from all channels of the continuum described by (25) is a necessary and sufficient condition for a resonance state to be a BIC, i.e. a state with vanishing decay width $\Gamma_{\lambda_0} = 0$. The wave function of such a BIC is, according to (12), eigenfunction of H_{eff} and, consequently, localized.

In [33], the advantage of the FPO method as compared to the N -level Friedrichs model [2] in studying BICs for unstable multilevel systems is discussed. In contrast to the coupling

matrix elements (25), the form factors $\langle \xi_C^E | V | \phi_{n_0}^B \rangle$ and $\langle \phi_{n_0}^B | V | \xi_C^E \rangle$ considered in [2] contain the basic wave functions ϕ_n^B of the Hamiltonian H_B of the closed system. Since the eigenfunctions ϕ_λ of H_{eff} can be represented as $\phi_\lambda = \sum a_{\lambda, \lambda'} \phi_{\lambda'}^B$ with complex coefficients $a_{\lambda, \lambda'}$ [and $\phi_\lambda^B = \sum b_{\lambda n} \phi_n^B$ with real $b_{\lambda n}$ and the basic wave functions ϕ_n^B of discrete states defining H_0 according to (2)], a sum of individual form factors vanishes at the position of a BIC according to (25),

$$\sum_{\lambda'} a_{\lambda_0, \lambda'} \langle \xi_C^E | V | \phi_{\lambda'}^B \rangle \rightarrow 0. \quad (26)$$

This equation is nonlinear since the coefficients $a_{\lambda_0, \lambda'}$ depend on the coupling strength. It is therefore difficult to obtain a general solution for the multilevel case in the framework of the Friedrichs model.

By using the FPO technique and demanding a vanishing decay width Γ_{λ_0} , the multilevel problem can be solved. Examples are given in [14] for atoms and in [31] for quantum dots. At sufficiently small coupling strength v , the condition $\Gamma_{\lambda_0} = 0$ can be fulfilled only when the spectrum of the closed system (described by H_B) is degenerated. Concrete examples of BICs investigated recently analytically with the postulation $\Gamma_{\lambda_0} = 0$ for a BIC are studies on open quantum billiards with variable shape [32]. The relation of the BICs to the avoided level crossing phenomenon can be seen in all examples. The scattering phase jumps by π in approaching the BIC in spite of the fact that no resonance can be seen in the cross section [31].

By means of the complex eigenvalues z_λ of H_{eff} , the appearance of a BIC can be traced as a function of a certain control parameter X , i.e. by controlling the trajectories $E_\lambda(X)$ and $\Gamma_\lambda(X)$. The BIC appears at the point $X = X_0$ where $\Gamma_\lambda(X_0) = 0$. It is even possible to consider the vicinity of the BIC including the cases when $\Gamma_\lambda(X')$ is always different from zero and $\Gamma_\lambda(X'_0)$ corresponds to the minimum of $\Gamma_\lambda(X')$ with a small but nonvanishing value $\Gamma_\lambda(X'_0) \approx 0$. This feature of the FPO technique is invaluable for applications since the stabilization of the system (caused by the vanishing width Γ_λ) must be known not only at the single point X_0 but also in its vicinity (where $\Gamma_\lambda > 0$, but small) in order to estimate the possibility of an experimental observation.

Examples of $\Gamma_\lambda(X)$ trajectories with $\Gamma_\lambda(X_0) = 0$ as well as with $\Gamma_\lambda(X'_0) \approx 0$ are studied on the basis of the FPO method for concrete systems, see [14] for atoms and [31] for quantum dots. In both cases, the trajectories depend strongly on energy near X_0 and X'_0 , respectively.

This energy dependence is caused by widths bifurcation, i.e. by the avoided level crossing phenomenon. The interaction of the resonance states via the continuum (described by the complex non-diagonal matrix elements of the second term of H_{eff}) plays an important role. The interplay between their real and imaginary parts makes possible the appearance of BICs at finite (physical) values of the coupling strength [14]. Every BIC appears together with at least one other state whose width is enhanced (due to the widths bifurcation) around $X = X_0$ and whose energy is, generally, close to E_{λ_0} . In the considered cases, the condition for the exact appearance of a (\mathcal{T} symmetric) BIC is space reflection symmetry (\mathcal{P} symmetry) of the system. Violation of \mathcal{P} symmetry leads to $\Gamma_\lambda(X'_0)$ small but different from zero, i.e. to violation of time reflection symmetry (violation of \mathcal{T} symmetry).

The BICs, being \mathcal{PT} symmetric states, are the result of widths bifurcation, i.e. finally of the existence of branch points in the complex plane. The branch points appear as families of trajectories when considered as a function of a parameter, as shown in numerical studies for double quantum dots [10]. As a consequence, also the BICs appear as families of trajectories by controlling the system by means of parameters. Besides the trajectories of the \mathcal{PT} symmetric BICs there are always trajectories of states whose widths are large (widths bifurcation). These states are \mathcal{P} symmetric, however \mathcal{T} symmetry is broken as discussed in Sects. 3 and 4.

It is interesting to remark that, using the condition of \mathcal{PT} symmetry, Bender et al. obtained classes of non-Hermitian Hamilton operators whose spectra are real and positive [34]. Recently, these solutions and their consequences for physical processes are discussed intensively in the literature, see e.g. the 2007 Workshop on Pseudo Hermitian Hamiltonians in Quantum Physics [35].

VI. SPECTROSCOPIC REORDERING PHENOMENA IN THE OVERLAPPING REGIME

An unsolved problem in standard quantum mechanics is the description of the crossover from the regime with weak coupling between discrete and continuous states to that with strong coupling between them. For example, an interpolation procedure between the limiting cases with isolated resonances at low level density and narrow resonances at high level density (superposed by a smooth background term) is introduced in [37] for the transmission

through a quantum dot. In contrast to such an interpolation procedure, the crossover can be described in the FPO formalism by means of the phase rigidity r_λ that is reduced for many states λ in this regime.

Let us consider the one-channel case, $C = 1$, and $\Psi_C^E \rightarrow \hat{\Psi}^E$ in the interior of the system. The right and left wave functions follow from (13) and (14) with $d_\lambda^E = c_\lambda^{E*}$ when excitation and decay of the state λ occur via the same mechanism. Therefore the $\hat{\Psi}^E$ can be normalized,

$$\langle \hat{\Psi}^{E,\text{left}} | \hat{\Psi}^{E,\text{right}} \rangle = \sum_{\lambda\lambda'} c_\lambda^{E*} c_{\lambda'}^E \langle \phi_\lambda^* | \phi_{\lambda'} \rangle = \sum_\lambda |c_\lambda^E|^2 \equiv 1. \quad (27)$$

The normalization has to be done separately at every energy E due to the explicit energy dependence of the c_λ^E . Moreover,

$$\langle \hat{\Psi}^{E,\text{left}*} | \hat{\Psi}^{E,\text{right}} \rangle = \sum_{\lambda\lambda'} c_\lambda^E c_{\lambda'}^E \langle \phi_\lambda | \phi_{\lambda'} \rangle = \sum_\lambda (c_\lambda^E)^2 A_\lambda \quad (28)$$

due to $B_\lambda^{\lambda'} = -B_{\lambda'}^\lambda$, see (9). A_λ is a real number [4]. From (27) and (28) follows

$$\frac{\langle \hat{\Psi}^{E*} | \hat{\Psi}^E \rangle}{\langle \hat{\Psi}^E | \hat{\Psi}^E \rangle} = \sum_\lambda (c_\lambda^E)^2 A_\lambda = \sum_\lambda \frac{(c_\lambda^E)^2}{r_\lambda}, \quad (29)$$

where the definition (10) for r_λ is used. Then the phase rigidity ρ of the wavefunctions $\hat{\Psi}^E$ may be defined by

$$\rho = e^{2i\theta} \sum_\lambda \frac{\text{Re}[(c_\lambda^E)^2]}{r_\lambda} = e^{2i\theta} \sum_\lambda \frac{1}{r_\lambda} \left([\text{Re}(c_\lambda^E)]^2 - [\text{Im}(c_\lambda^E)]^2 \right) \quad (30)$$

in analogy to (10). The value ρ corresponds to a rotation of $\hat{\Psi}^E$ by θ corresponding to the ratio between its real and imaginary parts. In spite of the complicated structure of ρ , it holds $1 \geq \rho \geq 0$, see [8]. The value ρ is uniquely determined by the spectroscopic properties of the system that are expressed by the coupling coefficients to the environment and the level density, or by the positions and widths of the resonance states and the phase rigidities r_λ . Eq. (30) shows the relation between ρ and the r_λ .

In [8, 25], the amplitude of the transmission through a quantum dot is considered in the framework of the S matrix theory,

$$t = -2\pi i \sum_\lambda \frac{\langle \xi_L^E | V | \phi_\lambda \rangle \langle \phi_\lambda^* | V | \xi_R^E \rangle}{E - z_\lambda}. \quad (31)$$

Here, the eigenvalues z_λ and eigenfunctions ϕ_λ of H_{eff} are involved with their full energy dependence, see Sect. 2.

For $\rho = 1$ and well isolated resonance states, the transmission amplitude (31) repeats the resonance structure of (11) of the wave function Ψ_C^E . The transmission peaks appear at the positions $E_\lambda \equiv \text{Re}(z_\lambda)|_{E=E_\lambda} \approx E_\lambda^B$ of the resonance states. An analogous result holds when there is a nonvanishing background term additional to the resonance term (31) of the transmission amplitude. The time scale corresponding to this so-called *direct* part of the transmission is, generally, well separated from that corresponding to the resonance part described by (31) [38]. Mostly, the resonances are narrow and well separated from one another. They appear as Fano resonances [40] on the smooth background. Due to the different time scales of the resonance and direct processes, it is $|\rho| \approx 1$ also in this case.

The situation is another one when the resonances overlap. In the overlapping regime, the resonance states avoid crossings with neighbored resonance states. In this case

$$\Gamma_\lambda < 4\pi \langle \xi_C^E | V | \phi_\lambda \rangle \langle \phi_\lambda^* | V | \xi_C^E \rangle \quad (32)$$

holds even when there is only one channel in each of the two identical leads. The relation (32) differs from

$$\Gamma_\lambda = 4\pi \langle \xi_C^E | V | \phi_\lambda \rangle \langle \phi_\lambda^* | V | \xi_C^E \rangle \quad (33)$$

for isolated resonances due to the biorthogonality of the eigenfunctions ϕ_λ that can not be neglected in the regime of overlapping resonances [4]. Therefore, the contribution of the state λ to $t_{(E \rightarrow E_\lambda)}$ is larger than 1. The unitarity condition will be fulfilled, nevertheless, due to interferences and the possibility to rotate the ϕ_λ , i.e. due to the non-rigidity of the phases of the wave functions ϕ_λ . As a consequence, the transmission in the overlapping regime does not show a resonance structure. Instead, it might be nearly plateau-like, for details see [8, 9, 25]. Let us rewrite therefore the transmission amplitude (31) by means of the wave function (13),

$$t = -2\pi i \langle \xi_{C'}^E | V | \hat{\Psi}_C^E \rangle \quad (34)$$

with $\hat{\Psi}_C^E$ being complex, in general. The advantage of this representation consists in the fact that it does not suggest the existence of resonance peaks in the transmission probability. Quite the contrary, the transmission is determined by the degree of alignment of the wave function $\hat{\Psi}_C^E$ with the propagating modes ξ_C^E in the leads, i.e. by the phase rigidity ρ . Nevertheless, the expressions (34) and (31) are fully equivalent.

The plateau-like structure of the transmission can not be obtained in standard quantum mechanics with fixed phases of the wave functions, $r_\lambda = 1$ and $\rho = 1$. It is generated by interference processes with account of the alignment of some of the resonance states to the scattering states ξ_C^E of the environment. At most, $\text{Re } \hat{\Psi}_C^E = \pm \text{Im } \hat{\Psi}_C^E$ (in the same manner as for the ξ_C^E). This case corresponds to $\rho = 0$. It will be reached when many resonance states are almost aligned with the ξ_C^E , and $\sum_\lambda \text{Re}[(c_{\lambda E})^2]/r_\lambda \approx 0$ according to (30).

The numerical results [25] obtained by using the tight-binding lattice Green function method [36] for the transmission through microwave cavities of different shape show exactly the features discussed above. In the weak-coupling regime as well as in the strong-coupling regime, the transmission shows a resonance structure as expected from the standard quantum mechanics. The only difference between the two cases is the appearance of a smooth background term in the strong-coupling regime which does not exist in the weak-coupling case, and the reduction of the number of resonance peaks by two (corresponding to the alignment of two resonance states each with one channel in each of the two identical attached leads).

In the crossover from the weak-coupling regime to the strong-coupling one, however, the calculated transmission is plateau-like instead of showing a resonance structure [25]. It is enhanced as compared to the transmission probability in the two borderline cases. In this regime, the resonance states overlap and spectroscopic reordering processes take place. Due to widths bifurcation, some of the resonance states become short-lived while other ones become trapped (long-lived). The enhancement of the transmission is caused by the short-lived states. Most interesting is the anticorrelation between transmission $|t|$ and phase rigidity $|\rho|$ which can be seen very clearly in all the numerical results obtained in [9, 25]. The behavior of the transmission in the crossover regime with overlapping resonance states does *not* correspond to the expectations of the standard quantum mechanics with rigid phases of the eigenfunctions of a Hermitian Hamilton operator, and decay widths obtained from poles of the S matrix, see e.g. [37]. Moreover, the transmission in the crossover regime is not only enhanced but it also outspeeds the transmission calculated in standard quantum mechanics. The reason is the formation of aligned (short-lived) resonance states in the vicinity of branch points in the complex plane.

The quantum brachistochrone problem of a physical system can be studied by considering the time needed for the transmission through the system from one of the attached leads to

another one [8]. The transmission time at a certain energy E is determined by the delay time, i.e. by the lifetime of the resonance states lying at this energy. The lifetime of a resonance state is bounded from below: it can not be smaller than allowing traveling through the system in accordance with traveling through the attached leads, i.e. the system may become transparent at most [41]. This lower bound can be reached in a system with the non-Hermitian Hamilton operator H_{eff} by aligning the wave functions of the system with those of the environment while such a possibility does not exist when H_{eff} is considered to be Hermitian.

VII. CONCLUSIONS

In the present paper it is shown that the FPO technique is a powerful method for the description of open quantum systems. The core of the method is the definition of two subspaces and the non-Hermitian symmetric Hamilton operator H_{eff} describing the localized states in one of the two subspaces. It guarantees a unified description of, on the one hand, discrete and resonance states and, on the other hand, resonance and decay processes. The spectroscopic information is received from the eigenvalues z_λ and eigenfunctions ϕ_λ of H_{eff} . Generally the eigenvalues are complex (with the exception of BICs and low-lying discrete states). In order to receive the spectroscopic information, the two subspaces must be defined in a meaningful manner. Observable values that are related to the S matrix and to the wave functions Ψ_C^E (being solutions of the Schrödinger equation with the Hermitian Hamilton operator H) are independent of the definition of the two subspaces as long as $P + Q = 1$ is fulfilled. The S matrix is always unitary, and it is not necessary to consider its poles.

The branch points in the complex plane determine the trajectories of the eigenvalues z_λ : level repulsion and widths bifurcation, respectively, appear in approaching them under different conditions. This phenomenon (avoided level crossing in the complex plane) causes the appearance of BICs in \mathcal{P} symmetric systems, i.e. of \mathcal{PT} symmetric solutions of the Schrödinger equation with non-Hermitian Hamilton operator. The BICs are localized at all times although no selection rule forbids their decay. Furthermore, the phases of the ϕ_λ are not rigid in the vicinity of the branch points. Due to this phenomenon, the eigenfunctions of H_{eff} may align to the scattering wave functions of the environment with the consequence that observable values are enhanced in the regime of overlapping resonances (where many

branch points exist).

The branch points in the complex plane influence also the properties of discrete states: the phenomenon of avoided crossing of discrete levels known for a long time, can be traced back to the branch points. Generally, the branch points in the complex plane introduce nonlinear effects into quantum mechanics. This fact allows, among others, to understand the well-known relation between avoided level crossings and quantum chaos.

Most interesting is that, due to the non-Hermiticity of H_{eff} , some problems of standard quantum mechanics are solved without any additional assumptions. For example, time asymmetry appears in a natural manner since H_{eff} contains a time operator. The results obtained by using the FPO formalism with the non-Hermitian Hamilton operator H_{eff} may be important for applications as well. By controlling the interplay between internal and external mixing of the resonance states by means of external parameters, the position of the localized BICs can be varied. Also the transmission through a system can be controlled parametrically: a system may become transparent in the crossover from the weak-coupling regime to the strong-coupling one.

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